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Orthogonally Additive Functions on Operator Algebras

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For a real or complex Hilbert space H, a general form of an even, orthogonally additive functional on $\mathbb{B}(H)$ is given.

INTRODUCTION

A celebrated theorem of Gleason (1975) states that a *Gleason measure*, i.e., a countably additive measure on the lattice of projections from $\mathbb{B}(H)$, with H separable and of dimension >2, extends uniquely to a positive linear functional on $\mathbb{B}(H)$. In this paper, we try to answer the following question: Are there any extensions of such a measure which are not linear, but preserve its orthogonal additivity? What do they look like? Of course, to be able to formulate the problem properly we need an orthogonality relation not only for projections, but also for operators. One natural definition suggests itself—two operators are orthogonal if the closures of their ranges are orthogonal.

The investigation of functions defined on various linear spaces and additive on orthogonal elements (with a suitably chosen orthogonality relation) has been a subject of many papers [see, e.g., references in Rosiński and Woyczyński (1977)]. It should be noted that $\mathbb{B}(H)$ (or, rather, its self-adjoint part) with the proposed notion of orthogonality is not, in general, an orthogonality vector space in the sense of Gudder and Strawther (1975), so that our results are not implied by theirs.

It is clear that an orthogonally additive functional on an operator algebra can be decomposed into its even and odd parts. In Theorem 2.3, which is the main result of this paper, we give a general form of an *even*, orthogonally

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additive functional on $\mathbb{B}(H)$. Using this result, we easily obtain the uniqueness of an even, orthogonally additive extension of a Gleason measure (see Corollary 2.7). As a byproduct, we obtain some (hopefully) new results on orthogonally additive functions on a Hilbert space, in both real and complex cases.

1. ORTHOGONALLY ADDITIVE FUNCTIONS ON A HILBERT SPACE

In the sequel, H denotes a real or complex Hilbert space. We shall always assume that $\dim H \geq 2$.

Definition 1.1. A function $f: H \to \mathbb{C}$ is called *orthogonally additive* (o.a. for short) if

$$\langle x, y\rangle = 0 \quad \text{implies} \quad f(x + y) = f(x) + f(y)$$

for any vectors $x, y \in H$.

Definition 1.2. A function $f: H \to \mathbb{C}$ is called *hemicontinuous* if

$$f(\lambda x) \to f(x) \qquad \text{for} \quad \lambda \to 1, \quad \lambda \in \mathbb{R}$$

for any vector $x \in H$.

The following fundamental result is well known.

Theorem 1.3. If an o.a. function f is hemicontinuous and real valued, then

$$f(x) = l(x) + \alpha\|x\|^2, \qquad x \in H \tag{1}$$

for some uniquely determined linear functional l on H and number $\alpha \in \mathbb{R}$.

This is a special case of Lemma 2.1 (and also Corollaries 2.3 and 2.4) in Gudder and Strawther (1975). Lemma 1 in Rosiński and Woyczyński (1977) for the Banach space $E = \mathbb{R}$ and the finite-dimensional Hilbert space $L^2(T, \Sigma, \mu)$ also reduces to our Theorem 1.3.

We shall prove the following result.

Theorem 1.4. Let H be a Hilbert space over $\mathbb{F}$ of dimension $\dim H \geq 2$.
(a) If $\mathbb{F} = \mathbb{R}$, then each real o.a. function on H, satisfying the condition

$$\sup\{|f(x)|; \|x\| \leq 1\} < \infty \tag{2}$$

is of the form

$$f(x) = \langle x, y_0\rangle + \alpha\|x\|^2, \qquad x \in H$$

for uniquely determined $y_0 \in H$ and $\alpha \in \mathbb{R}$.

(b) If $\mathbb{F} = \mathbb{C}$, then each complex o.a. function on H satisfying (2) is of the form

$$f(x) = \langle x, y_1 \rangle + \langle y_2, x \rangle + \alpha \|x\|^2, \qquad x \in H$$

for uniquely determined $y_1, y_2 \in H$, $\alpha \in \mathbb{C}$.

We begin with a few simple lemmas.

Lemma 1.5. If an o.a. function $f: H \to \mathbb{C}$ satisfies (2), then f is continuous (with respect to the norm) at $0 \in H$.

Proof. If $x, y \in H$, $\|x\| = \|y\|$ and $x \perp y$, then

$$\begin{aligned} f(x) + f(-x) &= f((x-y)/2) + f((x+y)/2) \\ &\quad + f((-x+y)/2) + f((-x-y)/2) \\ &= f(y) + f(-y) \end{aligned}$$

Consequently,

$$f(2x) = f(x+y) + f(x-y) = 3f(x) + f(-x)$$

and, by induction,

$$\begin{aligned} f(2^k x) + f(-2^k x) &= 2^{2k}(f(x) + f(-x)) \\ f(2^k x) - f(-2^k x) &= 2^k(f(x) - f(-x)) \end{aligned} \tag{3}$$

for each $k \in \mathbb{N}$.

Fix now $\epsilon > 0$. If, for any $k \in \mathbb{N}$, there exists $x \in H$ such that $\|x\| < 2^{-k}$ and $|f(x)| > \epsilon$, then

$$|f(x) + f(-x)| > \epsilon \tag{4}$$

or

$$|f(x) - f(-x)| > \epsilon \tag{5}$$

Inequality (4) implies, by (3),

$$\max(|f(2^k x)|, |f(-2^k x)|) \geq 1/2|f(2^k x) + f(-2^k x)| > 2^{2k-1}\epsilon$$

whereas (5) implies

$$\max(|f(2^k x)|, |f(-2^k x)|) \geq 1/2||f(2^k x) - f(-2^k x)| > 2^{k-1}\epsilon$$

Since $k \in \mathbb{N}$ was arbitrary, both cases lead to a violation of (2). ∎

Lemma 1.6. If an o.a. function $f: H \to \mathbb{C}$ is continuous at $0 \in H$, then it is hemicontinuous.

The proof does not differ from that of Corollary 2.4 in Gudder and Strawther (1975).

When H is a complex Hilbert space, by its real Hilbert subspace we mean a closed $\mathbb{R}$-linear subspace $K \subset H$ such that, for any $x, y \in K$, $\langle x, y\rangle \in \mathbb{R}$. Therefore, a real Hilbert subspace K with the induced scalar product is a real Hilbert space.

Lemma 1.7. Let H be a complex Hilbert space. If $f: H \to \mathbb{C}$ is continuous and $\mathbb{R}$-linear on each real Hilbert subspace of H, then

$$f(x) = \langle x, y_1\rangle + \langle y_2, x\rangle, \qquad x \in H$$

for some uniquely determined vectors $y_1, y_2 \in H$.

Proof. Note that any orthogonal system in H generates a real Hilbert subspace. Put, for $x \in H$,

$$g(x) = [f(x) - if(ix)]/2$$
$$h(x) = [\overline{f(x)} - \overline{if(ix)}]/2$$

Both g and h are continuous and $\mathbb{R}$-linear on real Hilbert subspaces of H, and both are $\mathbb{C}$-homogeneous on H. In fact, it is clear that $g(\gamma x) = \gamma g(x)$ for real or imaginary γ and $x \in H$. Take now any $\gamma \in \mathbb{C}$ and $x \in H$ and let e_1, $e_2 \in H$ be such that $x = e_1 + e_2$, $e_1 \perp e_2$, and $\|e_1\| = \|e_2\|$. Then

$$\begin{aligned} g(\gamma x) &= g(\Re\gamma e_1 + i\Im\gamma e_2) + g(i\Im\gamma e_1 + \Re\gamma e_2) \\ &= \Re\gamma g(e_1) + i\Im\gamma g(e_2) + i\Im\gamma g(e_1) + \Re\gamma g(e_2) \\ &= \gamma g(x) \end{aligned}$$

and a similar proof applies to h. Fix now a complete orthonormal system (e_k) in H. Take any $x \in H$. The properties of g on the real Hilbert subspace generated by vectors $\langle x, e_k\rangle e_k$ imply the summability of $(\langle x, e_k\rangle g(e_k))$. Thus, $(g(e_k))$ is square-summable and

$$g(x) = \sum g(\langle x, e_k\rangle e_k) = \sum \langle x, e_k\rangle g(e_k) = \langle x, \sum \overline{g(e_k)} e_k\rangle \qquad \text{for} \quad x \in H$$

Hence $g(x) = \langle x, y_1\rangle$ and similarly $h(x) = \langle x, y_2\rangle$ for some, obviously unique, vectors $y_1, y_2 \in H$. This yields the desired formula for $f(x) = g(x) + \overline{h(x)}$. ■

Proof of Theorem 1.4. Part (a) results directly from Lemmas 1.5 and 1.6 and Theorem 1.3, since (2) implies continuity of the functional f in representation (1). In order to prove (b), let us represent the function f as the sum of the even and odd parts $f = f^- + f^+$, where $f^{\pm}(x) = [f(x) \pm f(-x)]/2$. Fix now $e_0 \in H$, $\|e_0\| = 1$. Let e be an arbitrary vector in H with $\|e\| = 1$. Then there exist $e' \perp e$, $\langle e', e_0\rangle \in \mathbb{R}$, $\|e'\| = 1$, and real Hilbert subspaces K_1 and K_2, such that

$$e, e' \in K_1, \qquad e', e_0 \in K_2, \qquad \dim K_i \geq 2, \qquad i = 1, 2$$

Thus, in view of (a) and the evenness of f^+, we have

$$\Re f^+(\gamma e) = \alpha_e \gamma^2, \qquad \gamma \in \mathbb{R}$$

for some constant $\alpha_e \in \mathbb{R}$ depending on e. Since $\alpha_e = \alpha_{e'} = \alpha_{e_0}$, we obtain

$$\Re f^+(x) = \alpha_{e_0} \|x\|^2$$

Reasoning similarly for the imaginary part, we finally get

$$f^+(x) = \alpha \|x\|^2, \qquad x \in H$$

for a uniquely determined $\alpha \in \mathbb{C}$.

Also, by (a) and Lemma 1.7 we have

$$f^-(x) = \langle x, y_1 \rangle + \langle y_2, x \rangle, \qquad x \in H$$

for an odd function $f^-(x)$ satisfying (2). ∎

2. EVEN ORTHOGONALLY ADDITIVE FUNCTIONS ON $\mathbb{B}(H)$

Definition 2.1. We say that operators $A, B \in \mathbb{B}(H)$ are *(mutually) orthogonal* ($A \perp B$) if $B^*A = 0$.

Definition 2.2. A function ξ: $\mathbb{B}(H) \to \mathbb{C}$ is said to be *orthogonally additive* (o.a. for short) if:

(i) For each weakly summable family (A_i) of operators from $\mathbb{B}(H)$ satisfying $A_i \perp A_j$ for $i \neq j$, the family $(\xi(A_i))$ is summable and

$$\xi(\sum A_i) = \sum \xi(A_i)$$

(ii) $K := \sup\{ |\xi(A)| \mid \|A\| \leq 1\} < \infty$.

Theorem 2.3. If an o.a. function ξ on $\mathbb{B}(H)$ is even, i.e., if $\xi(A) = \xi(-A)$ for $A \in \mathbb{B}(H)$, then there exists a uniquely determined trace-class operator $M \in \mathbb{B}(H)$ such that

$$\xi(A) = \mathrm{tr}(MA^*A), \qquad A \in \mathbb{B}(H) \tag{6}$$

For any vectors $x, y \in H$ we denote by $\langle \cdot, x \rangle y$ the operator

$$H \ni z \mapsto \langle z, x \rangle y \tag{7}$$

This operator depends linearly on y and antilinearly on x, and $(\langle \cdot, x \rangle y)^* = \langle \cdot, y \rangle x$. The operator $\langle \cdot, e \rangle e$ is a one-dimensional projection for each $e \in H$, $\|e\| = 1$, and, for $x_1, x_2 \neq 0$,

$$\langle \cdot, x_1 \rangle y_1 \perp \langle \cdot, x_2 \rangle y_2 \quad \text{if and only if} \quad y_1 \perp y_2 \tag{8}$$

The essential part of the proof of Theorem 2.3 consists in proving formula (6) for operators A of the form (7). We shall break the proof into a series of lemmas.

Lemma 2.4. If the space H is complex (resp. real), then

$$\xi(\langle \cdot, x \rangle y) = \alpha(x)\|y\|^2, \qquad x, y \in H$$

for some (uniquely determined) complex (resp. real) function α on H.

Proof. For a fixed $x \in H$, the function

$$f(y) = \xi(\langle \cdot, x \rangle y), \qquad y \in H$$

is obviously even and, by (8), orthogonally additive. The use of Theorem 1.4 ends the proof. ∎

Lemma 2.5. Let $\alpha(\cdot)$ be the function from Lemma 2.4. For any $e_1, e_2 \in H$, $e_1 \perp e_2$, the equalities $\alpha(e_1) = \alpha(e_2) = \alpha(e_1 + e_2) = 0$ imply $\alpha(\beta_1 e_1 + \beta_2 e_2) = 0$ for any $\beta_1, \beta_2 \in \mathbb{R}$.

Proof. For $0 \neq \beta \in \mathbb{R}$, put $\delta = 1/(1 + \beta^2)^{1/2}$. Then, by the definition of $\alpha(\cdot)$,

$$\begin{aligned}
\alpha(\beta e_1 - \beta^{-1} e_2) &= \alpha(\beta e_1 - \beta^{-1} e_2) + \alpha(e_1 + e_2) \\
&= \xi(\langle \cdot, \beta e_1 - \beta^{-1} e_2 \rangle (\beta\delta e_1 + \delta e_2)) \\
&\quad + \xi(\langle \cdot, e_1 + e_2 \rangle (\delta e_1 - \beta\delta e_2)) \\
&= \xi(\langle \cdot, \beta e_1 - \beta^{-1} e_2 \rangle (\beta\delta e_1 + \delta e_2) \\
&\quad + (\langle \cdot, e_1 + e_2 \rangle (\delta e_1 - \beta\delta e_2)) \\
&= \xi(\delta^{-1} \langle \cdot, e_1 \rangle e_1 - (\beta\delta)^{-1} \langle \cdot, e_2 \rangle e_2) \\
&= \delta^{-2} \alpha(e_1) - (\beta\delta)^{-2} \alpha(e_2) \\
&= 0
\end{aligned}$$

Similarly, using the equality $\alpha(e_1 - e_2) = 0$, proved above, we show that $\alpha(\beta e_1 + \beta^{-1} e_2) = 0$. Now $\alpha(\gamma x) = \gamma^2 \alpha(x)$ for $\gamma \in \mathbb{R}$, $x \in H$, so that we can replace the coefficients β, β^{-1} with any pair β_1, β_2 of real coefficients, which ends the proof. ∎

Proposition 2.6. If ξ is an even, orthogonally additive function on $\mathbb{B}(H)$, then

$$\xi(P) = \mathrm{tr}(MP), \qquad P \in \mathcal{P}(H)$$

for a uniquely determined trace-class operator $M \in \mathbb{B}(H)$.

Proof. If $\dim H \geq 3$, the proposition follows immediately from a well-known generalization of the Gleason theorem for a signed Gleason measure, obtained originally by Šerstnev (n.d.) [see also Bunce and Wright (1992)].

Let $H = \mathbb{R}^2$. Choose an orthonormal basis $\{e_1, e_2\}$ of H. It is easy to find a self-adjoint $M \in \mathbb{B}(H)$ such that, for $\xi_1(A) = \xi(A) - \mathrm{tr}(MA^*A)$,

$$\xi_1(\langle\cdot, e_1\rangle e_1) = \xi_1(\langle\cdot, e_2\rangle e_2) = \xi_1(\langle\cdot, e_1 + e_2\rangle(e_1 + e_2)) = 0$$

By Lemmas 2.4 and 2.5, $\xi_1(\langle\cdot, x\rangle y) = \alpha(x)\|y\|^2$ with $\alpha = 0$, so that $\xi_1 = 0$.

Let now $H = \mathbb{C}^2$. As before, choose an orthonormal basis $\{e_1, e_2\}$ of H and put $\xi_1(A) = \xi(A) - \mathrm{tr}(MA^*A)$ for $A \in \mathbb{B}(H)$. It is easy to find $M \in \mathbb{B}(H)$ such that

$$\begin{aligned}\xi_1(\langle\cdot, e_1\rangle e_1) &= \xi_1(\langle\cdot, e_2\rangle e_2)\\ &= \xi_1(\langle\cdot, e_1 + e_2\rangle(e_1 + e_2))\\ &= \xi_1(\langle\cdot, e_1 + ie_2\rangle(e_1 + ie_2)) = 0\end{aligned}$$

By Lemma 2.4, $\xi_1(\langle\cdot, x\rangle y) = \alpha(x)\|y\|^2$, and

$$\alpha(e_1) = \alpha(e_2) = \alpha(e_1 + e_2) = \alpha(e_1 + ie_2) = 0$$

Moreover, $\alpha(\gamma x) = |\gamma|^2\alpha(x)$ for $\gamma \in \mathbb{C}$ and $x \in \mathbb{C}^2$. A repeated use of this formula together with Lemma 2.5 yields in turn $\alpha(-ie_1 + ie_2) = 0$, $\alpha(\overline{\gamma}e_1 + \gamma e_2) = 0$ for $\gamma \in \mathbb{C}$, $\alpha(e_1 + \gamma e_2) = 0$ for $\gamma \in \mathbb{C}$, $|\gamma| = 1$, $\alpha(e_1 + \gamma e_2) = 0$ for $\gamma \in \mathbb{C}$, and, finally, $\alpha(x) = 0$ for any $x \in \mathbb{C}^2$. Thus, $\xi_1 = 0$. The uniqueness of M is obvious in both the real and the complex case. ■

Proof of Theorem 2.3. Assume first that ξ is real (with H real or complex). Let (e_k) be an orthonormal basis of H. It is easy to see that the family $(\langle\cdot, A^*e_k\rangle A^*e_k)$ is weak operator summable with the sum equal to A^*A. Since the partial sums are uniformly bounded in norm (by $\|A\|^2$), they converge to A^*A σ-weakly as well. Using weak operator continuity of ξ, we get, by Lemma 2.4 and Proposition 2.6,

$$\begin{aligned}\xi(A) &= \xi(\sum_k \langle\cdot, e_k\rangle Ae_k)\\ &= \sum_k \xi(\langle\cdot, A^*e_k\rangle e_k)\\ &= \sum_k \alpha(A^*e_k)\\ &= \sum_k \|A^*e_k\|^2\alpha(A^*e_k/\|A^*e_k\|)\\ &= \sum_k \xi(\langle\cdot, A^*e_k/\|A^*e_k\|\rangle A^*e_k)\end{aligned}$$

$$= \sum_k \|A^*e_k\|^2 \xi(\langle \cdot , A^*e_k/\|A^*e_k\| \rangle A^*e_k/\|A^*e_k\|)$$

$$= \sum_k \|A^*e_k\|^2 \operatorname{tr}(M\langle \cdot , A^*e_k/\|A^*e_k\| \rangle A^*e_k/\|A^*e_k\|)$$

$$= \sum_k \operatorname{tr}(M\langle \cdot , A^*e_k\rangle A^*e_k)$$

$$= \operatorname{tr}(MA^*A)$$

(the terms with $A^*e_k = 0$ are all set to zero).

If ξ is complex, we decompose it into its real and imaginary parts. ∎

Corollary 2.7. Let μ be a countably orthogonally additive positive-valued function on the logic of all orthogonal projections in H (i.e., a Gleason measure on H), where H is a separable Hilbert space of dimension ≥ 3. Then there exists a unique extension of μ to an even, orthogonally additive functional on $\mathbb{B}(H)$ given by

$$f(A) = \operatorname{tr}(MA^*A)$$

for a uniquely determined positive trace-class operator M.

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